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New theories for convex fuzzy multi-objective optimization in a quotient space of fuzzy numbers

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Abstract. This paper aims to develop new theoretical foundations for solving convex multi-objective fuzzy optimization problems in a quotient space of fuzzy numbers. The primary objective is to extend the Karush-Kuhn-Tucker optimality conditions, originally designed for single-objective fuzzy optimization, to the multi-objective setting under convexity and differentiability assumptions. The methodology relies on defuzzification using midpoint functions, derived from α-cuts and Mareš cores, which transform fuzzy problems into classical equivalents. The proposed framework allows for the definition of Pareto, weak Pareto, and strong Pareto solutions in a fuzzy context. Key results include the necessary and sufficient Karush-Kuhn-Tucker optimality conditions. This approach bridges the gap between fuzzy mathematical theory and practical optimization techniques in uncertain environments.

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1. Introduction

Fuzzy multi-objective optimization is an advanced approach to classical optimization. It aims to solve problems involving multiple objectives while accounting for the uncertainty and vagueness of the data. It combines two key areas: multi-objective optimization, which finds optimal solutions for competing criteria, and fuzzy logic, which represents and manipulates uncertainty. The primary objective of fuzzy multi-objective optimization is to provide solutions that address the complexity and uncertain nature of real-world problems. It allows for the consideration of conflicting objectives. It finds compromises between them, offering a more realistic perspective for decision-making. This approach is used in many fields, including engineering, management, finance and planning, where decisions are often made in complex, uncertain environments. Then the notion of Pareto optimality is used to characterize the set of solutions for a multi-objective optimization problem. The search for Pareto solutions often relies on defuzzification techniques. These techniques transform a fuzzy optimization problem into a classical optimization problem. These techniques are used in both single- and multi-objective cases, regardless of the nature of the problem variables (e.g., continuous, discrete, binary), to facilitate the search for solutions. Several researchers have focused on different solution techniques for mono-objective fuzzy optimization (See, for example, [1, 2, 3, 4]).

However, the resolution technique that will be used for the remainder of our work is that of Nanxiang Yu and Dong Qiu [4]. Indeed, for a single-objective optimization problem of the type:

$$\begin{cases} \min \tilde{F}(t) = \tilde{F}(t_1, t_2, \cdots, t_n), \\ st \\ g_j(t) \leq 0, \; j = 1, 2, \cdots, m, \\ t \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

Let $\Omega = \{t \in \mathbb{R}^n : g_j(t) \leq 0, \, j = 1, 2, \cdots, m\}$ be the feasible set of problem (1.1).

For solving a problem of the type (1.1), the defuzzification technique used consists of determining the α-cuts of each equivalence class associated with its Mare$\check{s}$ core [5]. Then, we calculate the midpoint function associated with the objective function. Thus, by applying this defuzzification strategy to problem (1.1), we obtain the following problem:

$$\min \left(\mathtt{M}_{\tilde{F}(t)}(\alpha) \right), \tag{1.2}$$

where $\mathtt{M}_{\tilde{F}(t)}(\alpha)$ is the midpoint function of $\tilde{F}(t)$, $\alpha \in [0, 1]$.

This midpoint function, although it offers many advantages (See [4]) for solving mono-objective optimization problems, has not yet been tested in the multi-objective case. This motivates our work to contribute to the literature on the use of midpoint functions in multi-objective optimization.

Indeed, several researchers previously embarked on the quest for new techniques to solve fuzzy optimization problems. This would enable them to convert fuzzy optimization problems into classical optimization problems. Among these researchers were Bellman and Zadeh [6], who inspired the development of fuzzy optimization by proposing aggregation operators that combine fuzzy objectives and fuzzy decision spaces.

Motivated by this inspiration, numerous studies have been conducted on fuzzy optimization problems. Venkatesh et al. [7] propose a hybrid approach that combines machine learning and fuzzy optimization. In this approach, the predictions of the machine learning (ML) model are incorporated as fuzzy variables into a mixed-integer linear programming (MILP) model. A penalty function aligns the optimal decisions with the ML suggestions while ensuring operational feasibility. Vamarzani et al. [8] use a hybrid approach that combines robust fuzzy optimization and Chebyshev-type multi-choice goal programming to design a sustainable waste management network that balances economic, social, and environmental dimensions under uncertainty. Agrawal et al. [9] apply multi-criteria fuzzy optimization techniques, specifically F-SWARA, to determine the weights of financial ratios. They then use F-MOORA to rank Indian manufacturing firms based on their financial performance. Fernandez et al. Fernández et al. [10] model uncertain task durations using triangular fuzzy sets and incorporate them into a mixed-integer linear programming (MILP) model with delay penalties. The resulting fuzzy solution, computed using a MILP solver, provides a robust schedule for a single-machine construction project that tolerates imprecision while minimizing penalties. Wu [11] proposes a hybrid approach to fuzzy optimization that combines Shapley values from cooperative game theory with an evolutionary algorithm to explore non-dominated solutions and identify the most robust one. Sama and Some [12] propose a method based on the concept of a null set to solve mono-objective nonlinear fuzzy optimization problems. They transform the fuzzy problem into a deterministic, bi-objective problem and apply Karush-Kuhn-Tucker (KKT) optimality conditions to find an optimal solution. Then, they reconstruct the fuzzy solution using fuzzy algebraic operations. James and Jose [13] establish the Karush-Kuhn-Tucker optimality conditions for a fuzzy optimization problem in which the objective function is formulated using triangular q-rung orthopair fuzzy sets. The authors define the Hukuhara differentiability of these fuzzy functions and derive adapted KKT conditions to identify non-dominated solutions. Wu [2, 3] presented the KKT conditions for optimization problems with convex constraints and fuzzy objective functions in the class of all fuzzy numbers, considering the concepts of the Hausdorff metric and the Hukuhara difference. Chalco-Cano et al. [1] discussed the KKT optimality conditions for a class of fuzzy optimization problems using strongly generalized differentiable fuzzy functions. This concept of differentiability is more general for fuzzy applications than Hukuhara differentiability. These results in fuzzy optimization are based on well-known and widely used algebraic structures of fuzzy numbers, and the differentiability of fuzzy applications is based on the concept of Hukuhara difference. Operations in the set of fuzzy numbers are generally obtained by Zadeh's extension principle [14]. However, these operations may present some drawbacks for both theory and practical application. Specifically, no fuzzy number has an inverse element related to addition, whereas the inversion of addition is fundamental in arithmetic. Most researchers, such as [15, 16, 17, 18], propose different methods for constructing opposites of fuzzy numbers. But, these methods are innovative and have their own value, and most of them are abstract. In [5], Qiu et al. intuitively presented a method to find the inverse operation in the quotient space of fuzzy numbers based on the equivalence relation of Mare*š* [19, 20], which has the desired group properties for the addition operation [18, 21] on median functions. In [5], Qiu et al. further studied the differentiability properties of such functions in the quotient space of fuzzy numbers.

In the spirit of our predecessors, we propose a new theory to solve fuzzy multi-objective optimization problems. To do so, we will extend the KKT optimality conditions for single-objective fuzzy optimization problems in a quotient space of fuzzy numbers [4] to fuzzy, multi-objective, convex optimization problems in a quotient space of fuzzy numbers. Our plan is outlined below. Section 2 presents the fundamental concepts of our new theory. In Section 3, we formulate fuzzy, multi-objective programming problems with fuzzy-valued objective functions in a quotient space of fuzzy numbers. We also propose solution concepts and derive KKT conditions for these problems by introducing Lagrange multipliers.

2. Preliminaries

Definition 2.1 ([5, 22])**.** Let $f : [a, b] \to \mathbb{R}$ be a function. f is said to be of a *bounded variation*, if there exists a $C > 0$ such that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq C \tag{2.1}$$

for every partition $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ on $[a, b]$. The set of all functions of bounded variation on $[a, b]$ is denoted by $BV[a, b]$.

Definition 2.2 ([5, 22])**.** Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation. Then the *total variation* of f on $[a, b]$, denoted by $V_a^b(f)$, is defined by

$$V_a^b(f) = \sup_p \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|, \tag{2.2}$$

where p represents all partitions of $[a, b]$.

Definition 2.3 ([3, 12, 23, 24, 25])**.** Let X be a universe set. Then a *fuzzy set* $\tilde{A}$ on X is defined as follows:

$$\tilde{A} = \Big\{ \Big(x, \mu_{\tilde{A}}(x) \Big), x \in X \Big\} \text{ with } \mu_{\tilde{A}}(x) : X \to [0, 1].$$

$\mu_{\tilde{A}}$ is called a *membership function* of the set $\tilde{A}$.

Definition 2.4 ([22, 12, 23, 24, 25])**.** Let $\tilde{A}$ be a fuzzy set in X and $\alpha \in [0, 1]$. Then the *α-cut* of the fuzzy set $\tilde{A}$ is a set denoted by $\tilde{A}^\alpha$ and is defined by :

$$\tilde{A}^\alpha = \Big\{ x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha \Big\}. \tag{2.3}$$

Definition 2.5 ([2])**.** A fuzzy subset $\tilde{A}$ is called a *fuzzy number*, when the following conditions are satisfied:

(1) all the α-cut of $\tilde{A}$ are non-empty for $0 \leq \alpha \leq 1$,
(2) all the α-cut of $\tilde{A}$ are closed intervals in $\mathbb{R}$,
(3) the support of $\tilde{A}$ is bounded.

Example 2.6 ([3])**.** Let $\tilde{A} = (a, b, c, d)$, with a, b, c and $d \in \mathbb{R}$. We will say that $\tilde{A}$ *is a trapezoidal fuzzy number*, if its membership function is given by:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a}{b-a} & \text{if} \quad a \leq x < b, \\ 1 & \text{if} \quad b \leq x \leq c, \\ \frac{d-x}{d-c} & \text{if} \quad c < x \leq d, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Its α-cut has the following simplified form:

$$\tilde{A}^\alpha = [\alpha(b-a)+a, \alpha(c-d)+d] \quad \forall \alpha \in [0,1]. \tag{2.5}$$

Figure 1 is a graphical representation of its membership function.

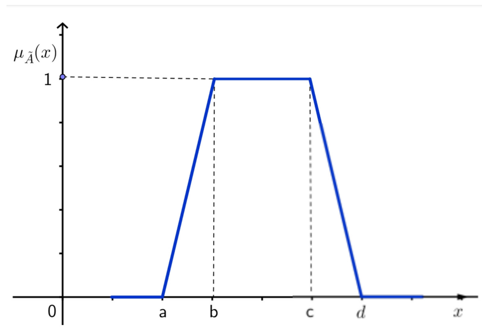

FIGURE 1. Membership function of a trapezoidal fuzzy number

Subsequently, we will denote by $\mathcal{N}(\mathbb{R})$ the set of all fuzzy numbers on $\mathbb{R}$. Then for $\tilde{x} \in \mathcal{N}(\mathbb{R})$ it is well known that the α-cut is a closed, bounded interval of $\mathbb{R}$. Thus we write $[\tilde{x}]^\alpha = [\tilde{x}_L(\alpha), \tilde{x}_R(\alpha)]$ a non-empty closed bounded interval in $\mathbb{R}$ for $\alpha \in [0,1]$, where $\tilde{x}_L(\alpha)$ denotes the left endpoint of $[\tilde{x}]^\alpha$ and $\tilde{x}_R(\alpha)$ denotes the right endpoint of $[\tilde{x}]^\alpha$. These are functions of α.

Definition 2.7 ([4, 19, 20, 26])**.** We say that a fuzzy number $\tilde{s} \in \mathcal{N}(\mathbb{R})$ is *symmetric*, if $\tilde{s} \tilde{=} -\tilde{s}$. In other words:

$$\forall x \in \mathbb{R}, \qquad \mu_{\tilde{s}}(x) = \mu_{\tilde{s}}(-x). \tag{2.6}$$

We denote the set of all symmetric fuzzy numbers by $\mathcal{S}$.

Definition 2.8 ([5, 19, 27])**.** Let $\tilde{x}$, $\tilde{y} \in \mathcal{N}(\mathbb{R})$, and we say that $\tilde{x}$ *is equivalent to* $\tilde{y}$, if there exist two symmetric fuzzy numbers $\tilde{s_1}$,$\tilde{s_2} \in \mathcal{S}$ such that $\tilde{x} + \tilde{s_1} \tilde{=} \tilde{y} + \tilde{s_2}$ and then, we denote this by $\tilde{x} \sim \tilde{y}$.

It is easy to verify that the equivalence relation defined above is reflexive, symmetric, and transitive [26]. Let denote the fuzzy number equivalence class containing the element and denote the set of all fuzzy number equivalence classes by $\mathcal{N}(\mathbb{R})/\mathcal{S}$.

Definition 2.9 ([17, 21, 28, 27])**.** For a fuzzy number $\tilde{x} \in \mathcal{N}(\mathbb{R})$, we define a function $\tilde{x}_{\mathtt{M}} : [0,1] \longrightarrow \mathbb{R}$ by assigning the midpoint of each α-level set to $\tilde{x}_{\mathtt{M}}(\alpha)$ for all $\alpha \in [0,1]$, i.e.,

$$\tilde{x}_{\mathtt{M}}(\alpha) = \frac{\tilde{x_L}(\alpha) + \tilde{x_R}(\alpha)}{2}. \tag{2.7}$$

Then the function $\tilde{x}_{\mathtt{M}} : [0,1] \longrightarrow \mathbb{R}$ will be called the *midpoint function* of the fuzzy number $\tilde{x}$.

Lemma 2.10 ([4, 5]). *For any $\tilde{x} \in \mathcal{N}(\mathbb{R})$, the midpoint function $\tilde{x}_{\mathtt{M}}$ is continuous from the right at* 0 *and continuous from the left on* $[0,1]$. *Furthermore, it is a function of bounded variation on* $[0,1]$.

Definition 2.11 ([4, 20]). Let $\tilde{x} \in \mathcal{N}(\mathbb{R})$ and let $\hat{x}$ be a fuzzy number such that $\tilde{x} = \hat{x} + \tilde{s}$ for some $\tilde{s} \in \mathcal{S}$, and if $\hat{x} = \tilde{y} + \tilde{s_1}$ for some $\tilde{y} \in \mathcal{N}(\mathbb{R})$ and $\tilde{s_1} \in \mathcal{S}$, then $\tilde{s_1} = \tilde{0}$. Then the fuzzy number $\hat{x}$ will be called the $Mare\check{s}$ *core* of the fuzzy number $\tilde{x}$.

Definition 2.12 ([4, 27]). Let $\langle\tilde{x}\rangle \in \mathcal{N}(\mathbb{R})/\mathcal{S}$ and we define the midpoint function $\mathtt{M}_{\langle\tilde{x}\rangle} : [0,1] \to \mathbb{R}$ by

$$\mathtt{M}_{\langle\tilde{x}\rangle}(\alpha) = \hat{x}_{\mathtt{M}}(\alpha) \ for\ all\ \alpha \in [0,1], \tag{2.8}$$

where $\hat{x}$ is the $Mare\check{s}$ core of $\tilde{x}$.

Definition 2.13 (See [4, 5]). Let $\langle\tilde{x}\rangle, \langle\tilde{y}\rangle \in \mathcal{N}(\mathbb{R})/\mathcal{S}$ and we define the *sum*, the *product* and *scalar multiplication* of these two fuzzy number equivalence classes as a fuzzy equivalence class $\langle\tilde{z}\rangle \in \mathcal{N}(\mathbb{R})/\mathcal{S}$, which satisfies the condition:

$$\mathtt{M}_{\langle\tilde{x}\rangle}(\alpha) + \mathtt{M}_{\langle\tilde{y}\rangle}(\alpha) = \mathtt{M}_{\langle\tilde{z}\rangle}(\alpha), \tag{2.9}$$

for all $\alpha \in [0,1]$ and we denote this by

$$\langle\tilde{x}\rangle(+)\langle\tilde{y}\rangle \tilde{=} \langle\tilde{x}(+)\tilde{y}\rangle \tilde{=} \langle\tilde{z}\rangle. \tag{2.10}$$

$$\mathtt{M}_{\langle\tilde{x}\rangle}(\alpha) \cdot \mathtt{M}_{\langle\tilde{y}\rangle}(\alpha) = \mathtt{M}_{\langle\tilde{z}\rangle}(\alpha), \tag{2.11}$$

for all $\alpha \in [0,1]$ and we denote this by

$$\langle\tilde{x}\rangle(\cdot)\langle\tilde{y}\rangle \tilde{=} \langle\tilde{z}\rangle. \tag{2.12}$$

We define $\lambda(\cdot)\langle\tilde{x}\rangle \tilde{=} \lambda\langle\tilde{x}\rangle$, $\lambda \in \mathbb{R}$ by

$$\lambda\langle\tilde{x}\rangle \tilde{=} \langle\tilde{x}\rangle\lambda \tilde{=} \langle\lambda\tilde{x}\rangle. \tag{2.13}$$

It is obvious that $\mathtt{M}_{\lambda\langle\tilde{x}\rangle}(\alpha) = \mathtt{M}_{\langle\lambda\tilde{x}\rangle}(\alpha) = \lambda\mathtt{M}_{\langle\tilde{x}\rangle}(\alpha)$ for all $\alpha \in [0,1]$.

Definition 2.14 ([4]). Let $\langle\tilde{x}\rangle, \langle\tilde{y}\rangle \in \mathcal{N}(\mathbb{R})/\mathcal{S}$.

(i) $\langle\tilde{x}\rangle \tilde{\leq} \langle\tilde{y}\rangle$, if $\mathtt{M}_{\langle\tilde{x}\rangle}(\alpha) \leq \mathtt{M}_{\langle\tilde{y}\rangle}(\alpha)$ for all $\alpha \in [0,1]$.

(ii) $\langle\tilde{x}\rangle \tilde{<} \langle\tilde{y}\rangle$, if $\langle\tilde{x}\rangle \tilde{\leq} \langle\tilde{y}\rangle$ and there exists at least one $\alpha_0 \in [0,1]$ such that $\mathtt{M}_{\langle\tilde{x}\rangle}(\alpha_0) < \mathtt{M}_{\langle\tilde{y}\rangle}(\alpha_0)$.

(iii) If $\langle\tilde{x}\rangle \tilde{\leq} \langle\tilde{y}\rangle$ and $\langle\tilde{y}\rangle \tilde{\leq} \langle\tilde{x}\rangle$, then $\langle\tilde{x}\rangle \tilde{=} \langle\tilde{y}\rangle$.

Sometimes we may write $\langle\tilde{y}\rangle \tilde{\geq} \langle\tilde{x}\rangle$ instead of $\langle\tilde{x}\rangle \tilde{\leq} \langle\tilde{y}\rangle$ and write $\langle\tilde{y}\rangle \tilde{>} \langle\tilde{x}\rangle$ instead of $\langle\tilde{x}\rangle \tilde{<} \langle\tilde{y}\rangle$. Note that "$\tilde{\leq}$" is a partial order relation on $\mathcal{N}(\mathbb{R})/\mathcal{S}$

Definition 2.15 (See [4, 5]). Define
$d_{sup} : \mathcal{N}(\mathbb{R})/\mathcal{S} \times \mathcal{N}(\mathbb{R})/\mathcal{S} \longrightarrow \mathbb{R}^+ \cup \{0\}$ by

$$d_{\sup}(\langle\tilde{x}\rangle, \langle\tilde{y}\rangle) = \sup_{\alpha \in [0,1]} \left|\mathtt{M}_{\langle\tilde{x}\rangle}(\alpha) - \mathtt{M}_{\langle\tilde{y}\rangle}(\alpha)\right| \tag{2.14}$$

for all $\langle\tilde{x}\rangle, \langle\tilde{y}\rangle \in \mathcal{N}(\mathbb{R})/\mathcal{S}$. Then $(\mathcal{N}(\mathbb{R})/\mathcal{S}), d_{sup})$ is a metric space [5]

In this paper, we always suppose that the range of fuzzy mappings is the set of all fuzzy number equivalence classes.

Definition 2.16 ([27]). Let $\tilde{F} : T \to \mathcal{N}(\mathbb{R})/\mathcal{S}$ be a fuzzy mapping, where $T = [a, b] \subseteq \mathbb{R}$. Then $\tilde{F}$ is said to be *differentiable at* $t \in T$, if there exists an $\tilde{F}'(t) \in \mathcal{N}(\mathbb{R})/\mathcal{S}$ such that

$$\lim_{h\to 0} d_{\sup}\left(\frac{\tilde{F}(t+h)\,(-)\,\tilde{F}(t)}{h},\ \tilde{F}'(t)\right) = 0. \tag{2.15}$$

If $t = a(or\, b)$, then we consider only $h \to 0^+(or\, h \longrightarrow 0^-)$.

Lemma 2.17 ([27]). *$\tilde{F} : T \to \mathcal{N}(\mathbb{R})/\mathcal{S}$ is differentiable on T if and only if*

(1) $\mathtt{M}_{\tilde{F}(t)}(\alpha)$ *is differentiable with respect to $t \in T$, for all $\alpha \in [0, 1]$. That is, $\left(\frac{\partial}{\partial t}\mathtt{M}_{\tilde{F}(t)}(\alpha)\right)$ exists and is of bounded variation with respect to $\alpha \in [0, 1]$for all $t \in T$,*

(2) *the mappings $\mathtt{M}_{\tilde{F}(t)}(\alpha)$ are uniformly differentiable with the derivatives $\left(\frac{\partial}{\partial t}\mathtt{M}_{\tilde{F}(t)}(\alpha)\right)$, i.e., for each $t \in T$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\left|\frac{\mathtt{M}_{\tilde{F}(t+h)}(\alpha) - \mathtt{M}_{\tilde{F}(t)}(\alpha)}{h} - \frac{\partial}{\partial t}\mathtt{M}_{\tilde{F}(t)}(\alpha)\right| < \varepsilon \tag{2.16}$$

for all $|h| \in (0, \delta)$ and $\alpha \in [0, 1]$.

Definition 2.18 ([4]). Let $\langle\tilde{a}\rangle\tilde{=}(\langle\tilde{a_1}\rangle, \langle\tilde{a_2}\rangle, \cdots, \langle\tilde{a_n}\rangle)^T \in (\mathcal{N}(\mathbb{R})/\mathcal{S})^n$ and $t = (t_1, t_2, \cdots, t_n)^T \in \mathbb{R}^n$ be an n-dimensional fuzzy number equivalence class vector and n-dimensional real vector, respectively. We define their *product* $\langle\tilde{a}\rangle^T t$ as follows:

$$\langle\tilde{a}\rangle^T t\tilde{=}\sum_{i=1}^{n}\langle\tilde{a_i}\rangle t_i\tilde{=}\langle\tilde{a_1}\rangle t_1 + \langle\tilde{a_2}\rangle t_2 + \cdots + \langle\tilde{a_n}\rangle t_n, \tag{2.17}$$

which is a fuzzy number equivalence class.

Definition 2.19. Let $\tilde{F}_i : \mathbb{R}^n \to \mathcal{N}(\mathbb{R})/\mathcal{S}$, $i = 1, 2, \cdots, p$ and $\tilde{g}_j : \mathbb{R}^n \to \mathcal{N}(\mathbb{R})/\mathcal{S}$, $j = 1, 2, \cdots, m$, fuzzy-value functions. Then a *multi-objective optimization problem* in a quotient space of fuzzy numbers with fuzzy constraints is of the following form:

$$\begin{cases} \min \tilde{F}(t), \\ st \\ \tilde{g}(t)\tilde{\leq}\langle\tilde{0}\rangle, \\ t \in \mathbb{R}^n, \end{cases} \tag{2.18}$$

with $\tilde{F}(t)\tilde{=}\Big(\tilde{F}_1(t), \tilde{F}_2(t), \cdots, \tilde{F}_p(t)\Big)^T$, $\tilde{g}(t)\tilde{=}\Big(\tilde{g}_1(t), \tilde{g}_2(t), \cdots, \tilde{g}_m(t)\Big)^T$ and $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$.

Let $\chi = \{t \in \mathbb{R}^n : \tilde{g}_j(t)\tilde{\leq}\langle\tilde{0}\rangle,\ j = 1, 2, \ldots, m\}$ be the feasible domain of (2.18).

Definition 2.20. An admissible solution $t^* \in \chi$ is said to be $efficient$ (Pareto optimal), if there does not exist another $t \in \chi$ such that $\tilde{F}(t)$ dominates $\tilde{F}(t^*)$.

Definition 2.21. An admissible solution $t^* \in \chi$ is said to be *weakly efficient* (weakly Pareto optimal), if there does not exist another $t \in \chi$ such that $\tilde{F}(t)\tilde{<}\tilde{F}(t^*)$, i.e.,

$$\tilde{F}_i(t)\tilde{<}\tilde{F}_i(t^*),\ i = 1, \cdots, p.$$

Definition 2.22. An admissible solution $t^* \in \chi$ is said to be *strongly efficient* (strongly Pareto optimal), if there does not exist another $t \in \chi$, $t \neq t^*$ such that $\tilde{F}(t)\tilde{\leq}\tilde{F}(t^*)$.

Remark 2.23. Let χ_{fp}, χ_p, χ_{Fp} be the sets of weakly Pareto optimal, Pareto optimal, and strongly Pareto optimal solutions respectively. We have the following relation

$$\chi_{fp} \subseteq \chi_p \subseteq \chi_{Fp}.$$

Definition 2.24 ([4])**.** Let $\tilde{F} : \Omega \to \mathcal{N}(\mathbb{R})/\mathcal{S}$ be a fuzzy mapping, where Ω is nonempty convex subset in $\mathbb{R}^n$. $\tilde{F}$ is said to be *convex* on Ω, if for any $s, t \in \Omega$ and $\lambda \in [0,1]$, we always have $\tilde{F}(\lambda s + (1-\lambda)t)\tilde{\leq}\lambda\tilde{F}(s)\,(+)\,(1-\lambda)\tilde{F}(t)$. $\tilde{F}$ is said to be *concave*, if $-\tilde{F}$ is convex.

Theorem 2.25 ([4])**.** *Let $\tilde{F} : \Omega \to \mathcal{N}(\mathbb{R})/\mathcal{S}$ be a fuzzy mapping, where Ω is a nonempty convex subset $\mathbb{R}^n$. Then $\tilde{F}$ is convex on Ω if and only if $\mathtt{M}_{\tilde{F}(t)}(\alpha)$ is convex with respect to $t \in \Omega$ for all $\alpha \in [0,1]$.*

Proof. See the proof of Theorem 3.3 in [29]. □

Remark 2.26. Problem (2.18) is convex and differentiable if all the fuzzy-valued functions and the fuzzy-valued constraint functions are all convex and differentiable.

Definition 2.27 ([30])**.** Let $\tilde{F} : \Omega \to \mathcal{N}(\mathbb{R})/\mathcal{S}$ be a differentiable fuzzy mapping, where Ω is a nonempty convex subset in $\mathbb{R}^n$. $\tilde{F}$ is said to be *pseudoconvex* on Ω, if for any $s, t \in \Omega$ such that $\tilde{F}(s)\tilde{<}\tilde{F}(t)$, we always have

$$\nabla\tilde{F}(t)^T(s-t)\tilde{<}\langle\tilde{0}\rangle. \tag{2.19}$$

In the following, we will assume that problem (2.18) is convex and differentiable.

3. Results

Let us consider the problem (2.18). By determining the α-cuts of each equivalence class associated with its Mareš core, then, by calculating the midpoint functions associated with the objective functions and constraint functions, and by applying this strategy to problem (2.18) we obtain the following problem

$$\begin{cases} \min \mathtt{M}_{\tilde{F}_1(t)}(\alpha), \\ \min \mathtt{M}_{\tilde{F}_2(t)}(\alpha), \\ \quad\vdots \\ \min \mathtt{M}_{\tilde{F}_p(t)}(\alpha), \\ st \\ \mathtt{M}_{\tilde{g}_1(t)}(\alpha) \leq 0, \\ \mathtt{M}_{\tilde{g}_2(t)}(\alpha) \leq 0, \\ \quad\vdots \\ \mathtt{M}_{\tilde{g}_m(t)}(\alpha) \leq 0 \\ t \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

with $\mathtt{M}_{\tilde{F}_i(t)}(\alpha)$, $\mathtt{M}_{\tilde{g}_j(t)}(\alpha)$ respectively the midpoint functions of $\tilde{F}_i(t)$, $i = 1, 2, \cdots, p$ and $\tilde{g}_j(t)$, $j = 1, 2, \cdots, m$ with $\alpha \in [0, 1]$.

Let $\Phi = \{t \in \mathbb{R}^n : \mathtt{M}_{\tilde{g}_j(t)}(\alpha) \leq 0, j = 1, 2, \cdots, m\}$ be the feasible domain of problem (3.1).

We present the links that exist between the Pareto optimal solutions of the problem (3.1) and the Pareto optimal solutions of the initial problem (2.18) in the form of the following theorems.

Theorem 3.1. *If t^* is a Pareto optimal solution of problem* (3.1) *for all $\alpha \in [0, 1]$ then t^* is a Pareto optimal solution of problem* (2.18)*.*

Proof. Let $t^* \in \Phi$ be a Pareto optimal solution of (3.1).

Assume that t^* is not a Pareto optimal solution of problem (2.18). Then there exists $\bar{t} \in \chi$ such that $\tilde{F}_i(\bar{t}) \tilde{\leq} \tilde{F}_i(t^*)$ $\forall i = 1, \cdots, p$ and $\tilde{F}_k(\bar{t}) \tilde{<} \tilde{F}_k(t^*)$ for at least one $k \in \{1, \cdots, p\}$. Switching to the midpoint functions, there exists $\alpha \in [0, 1]$ such that $\mathtt{M}_{\tilde{F}_i(\bar{t})}(\alpha) \leq \mathtt{M}_{\tilde{F}_i(t^*)}(\alpha)$ $\forall i = 1, \cdots, p$ and $\mathtt{M}_{\tilde{F}_k(\bar{t})}(\alpha) < \mathtt{M}_{\tilde{F}_k(t^*)}(\alpha)$ for at least one $k \in \{1, \cdots, p\}$. This contradicts the fact that t^* is a Pareto optimal solution of (3.1) for all $\alpha \in [0, 1]$. Thus t^* is a Pareto optimal solution of problem (2.18). $\square$

Theorem 3.2. *If t^* is a weakly Pareto optimal solution of problem* (3.1) *for $\alpha \in [0, 1]$, then t^* is a weakly Pareto optimal solution of problem* (2.18)*.*

Proof. Let $t^* \in \Phi$ be a weakly Pareto optimal solution of problem (3.1). Suppose t^* is not a weakly Pareto optimal solution of problem (2.18) . Then there exists $\bar{t} \in \chi$ such that $\tilde{F}_i(\bar{t}) \tilde{\leq} \tilde{F}_i(t^*)$, $\forall i = 1, \cdots, p$. Switching to midpoint functions, there exists $\alpha \in [0, 1]$ such that $\mathtt{M}_{\tilde{F}_i(\bar{t})}(\alpha) \leq \mathtt{M}_{\tilde{F}_i(t^*)}(\alpha)$, $\forall i = 1, \cdots, p$. With the strict inequality, that is $\mathtt{M}_{\tilde{F}_i(\bar{t})}(\alpha) < \mathtt{M}_{\tilde{F}_i(t^*)}(\alpha)$. In particular, this relationship holds for $\alpha^* \in [0, 1]$, which contradicts the fact that t^* is a weakly Pareto optimal solution of problem (3.1) for $\alpha^* \in [0, 1]$. $\square$

In the following, we will provide the Karush-Kuhn-Tucker optimality conditions for Pareto optimal solutions.

Remark 3.3. Considering Lemma 2.17 and Theorem 2.25, problem (3.1) is convex and differentiable if and only if problem (2.18) is convex and differentiable.

Definition 3.4. Let the constraint functions $\mathtt{M}_{\tilde{g}_j(t)}(\alpha) \leq 0$ of problem (3.1) be continuously differentiable at t^*. The problem satisfies the *Kuhn-Tucker constraint qualification* at t^* if for any $d \in \mathbb{R}^n$ such that $\mathtt{M}_{\nabla \tilde{g}_j(t^*)}(\alpha)^T d \leq 0$ for all $j \in J(t^*)$ (with J be the set for the active constraints of problem (3.1)), there exists a function $a : [0, 1] \to \mathbb{R}^n$ which is continuously differentiable at 0 and some real scalar $\beta > 0$ such that $a(0) = t^*$, $\quad \mathtt{M}_{g(a(t))}(\alpha) \leq 0 \quad$ for all $0 \leq t \leq 1 \quad$ and $\quad a'(0) = \beta d$.

Theorem 3.5. *Let Φ be the convex feasible set and $t^* \in \Phi$ be a feasible solution of problem* (3.1)*. Suppose that the real-valued constraint functions $M\tilde{g}_j(t)(\alpha)$ and fuzzy-valued objective functions $\tilde{F}_i$ are convex on $\mathbb{R}^n$ and continuously differentiable at t^* for all $j = 1, 2, \cdots, m$ and $i = 1, 2, \cdots, p$. If there exist positive real-valued functions* (positive Lagrange function multipliers) $\lambda_i(\alpha)$ *defined on* $[0, 1]$ *for*

$i = 1, 2, \cdots, p$ *and nonnegative real-valued functions* $\mu_j(\alpha)$ (nonnegative Lagrange function multipliers) *defined on* $[0,1]$ *such that*

(i) $\sum_{i=1}^{p} \lambda_i(\alpha) \mathtt{M}_{\nabla \tilde{F}_i(t^*)}(\alpha) + \sum_{j=1}^{m} \mu_j(\alpha) \mathtt{M}_{\nabla \tilde{g}_j(t^*)}(\alpha) = 0$, *for all* $\alpha \in [0,1]$,

(ii) $\mu_j(\alpha) \mathtt{M}_{\tilde{g}_j(t^*)}(\alpha) = 0$, *for all* $\alpha \in [0,1]$ *and for all* $j = 1, 2, \ldots, m$,

then t^* *is a Pareto optimal solution of problem* (2.18).

Proof. We are going to prove this result by contradiction. Suppose that t^* is not a Pareto optimal solution. Then there exist $\bar{t}(\neq t^*) \in \Omega$ such that $\tilde{F}_i(\bar{t}) \tilde{<} \tilde{F}_i(t^*)$, i.e., there is $\alpha^* \in [0,1]$ such that

$$\mathtt{M}_{\tilde{F}_i(\bar{t})}(\alpha^*) < \mathtt{M}_{\tilde{F}_i(t^*)}(\alpha^*). \tag{3.2}$$

We now define a real-valued function

$$f(t) = \sum_{i=1}^{p} \lambda_i(\alpha) \mathtt{M}_{\tilde{F}_i(t)}(\alpha^*). \tag{3.3}$$

Since the fuzzy mappings $\tilde{F}_i$, $i = 1, 2, \cdots, p$ are convex on $\mathbb{R}^n$ and continuously differentiable at t^*, by Theorem 2.25 and Lemma 2.17 we see that f is also convex on $\mathbb{R}^n$ and continuously differentiable at t^*.

Furtheremore, we have $\nabla f(t) = \sum_{i=1}^{p} \lambda_i(\alpha) \nabla \mathtt{M}_{\tilde{F}_i(t)}(\alpha^*) = \sum_{i=1}^{p} \lambda_i(\alpha) \mathtt{M}_{\nabla \tilde{F}_i(t)}(\alpha^*)$. From (3.3) and in accordance with conditions (i) and (ii) of this theorem, we derive the following two new conditions for any fixed $\alpha^* \in [0,1]$:

(a) $\nabla f(t^*) + \sum_{j=1}^{m} \mu_j(\alpha^*) \mathtt{M}_{\nabla \tilde{g}_j(t^*)}(\alpha^*) = 0$,

(b) $\mu_j(\alpha^*) \mathtt{M}_{\tilde{g}_j(t^*)}(\alpha^*)$, for all $j = 1, 2, \ldots, m$.

If (a) and (b) have the same conditions as Theorem 25 in [4], then we obtain that t^* is an optimal solution of the real-valued objective function f, i.e.,

$$f(t^*) < f(\bar{t}) \tag{3.4}$$

for any $\bar{t} \in \Phi(\bar{t} \neq t^*)$. From (3.2) and (3.3), we see that $f(\bar{t}) < f(t^*)$ which contradicts inequality (3.4). Thus we conclude that t^* is indeed a Pareto optimal solution of problem (2.18). □

Lemma 3.6 ([4]). *Let* $\Phi = \{t \in \mathbb{R}^n : \mathtt{M}_{\tilde{g}_j(t)}(\alpha) \leq 0, j = 1, 2, \cdots, m\}$ *be a feasible set and* $t^* \in \Omega$. *Assume that* $\mathtt{M}_{\tilde{g}_j}$ *are differentiable at* t^* *for all* $j = 1, 2, \cdots, m$. *Let* $J = \{j : g_j(t) = 0\}$ *be the index set for the active constraints. Then we have*

$$D \subseteq \{d \in \mathbb{R}^n : \mathtt{M}_{\nabla \tilde{g}_j(t^*)}(\alpha)^T d \leq 0 \ \forall j \in J\}, \tag{3.5}$$

where D *is the cone of feasible directions of* Φ *at* t^* *defined by*

$$D = \{d \in \mathbb{R}^n : d \neq 0, \text{ there exists a } \delta > 0 \text{ such that } t^* + \eta d \in \Phi \ \forall \eta \in (0, \delta)\}. \tag{3.6}$$

Lemma 3.7 ([4]). *Let* A *and* C *be two matrices. Exactly one of the following systems has a solution:*

- ***system I:*** $Ax \leq 0$, $Ax \neq 0$, $Cx \leq 0$ *for some* $x \in \mathbb{R}^n$,
- ***system II:*** $A^T \lambda + C^T u = 0$ *for some* (λ, u), $\lambda > 0$, $u \geq 0$.

Theorem 3.8. *Let Φ be the convex feasible set and $t^* \in \Phi$ be a feasible solution of problem (3.1). Suppose that the real-valued constraint functions $\mathtt{M}_{\tilde{g}_j(t)}(\alpha)$ are convex and continuously differentiable at t^* for $j = 1, 2, \cdots, m$ and the fuzzy-valued objective functions $\tilde{F}_i$ are continuously differentiable at t^* for $i = 1, 2, \cdots, p$. We additionally assume that each of the fuzzy-valued objective functions is either strictly pseudoconvex or convex at t^*. If there exist positive real-valued functions $\lambda_i(\alpha)$ defined on $[0, 1]$ for all $i = 1, 2, \cdots, p$ and nonnegative real-valued functions $\mu_j(\alpha)$ defined on $[0, 1]$ for and (Lagrange multipliers for $j = 1, 2, \cdots, m$ such that*

(i) $\displaystyle\sum_{i=1}^{p} \lambda_i(\alpha)\mathtt{M}_{\nabla\tilde{F}_i(t^*)}(\alpha) + \sum_{j=1}^{m} \mu_j(\alpha)\mathtt{M}_{\nabla\tilde{g}_j(t^*)}(\alpha) = 0$ *for all* $\alpha \in [0, 1]$,

(ii) $\mu_j(\alpha)\mathtt{M}_{\tilde{g}_j(t^*)}(\alpha) = 0$ *for all* $\alpha \in [0, 1]$ *and all* $j = 1, 2, \cdots, m$,

then t^* is a Pareto optimal solution of problem (2.18).

Proof. We know that each of the fuzzy-valued objective functions is either strictly pseudoconvex or convex at t^*. Then the functions $\mathtt{M}_{\tilde{F}_i(t^*)}(\alpha)$ are either strictly pseudoconvex or convex at t^* for all $\alpha \in [0, 1]$ and $i = 1, 2, \ldots, p$. Suppose that t^* is not a Pareto optimal solution of problem (2.18). Then there exists $\bar{t} \in \Phi$ ($\bar{t} \neq t^*$) such that $\tilde{F}_i(\bar{t}) \tilde{<} \tilde{F}_i(t^*)$. That is for $\alpha^* \in [0, 1]$, we have $\mathtt{M}_{\tilde{F}_i(\bar{t})}(\alpha^*) < \mathtt{M}_{\tilde{F}_i(t^*)}(\alpha^*)$. Since $\mathtt{M}_{\tilde{F}_i(t^*)}(\alpha^*)$ are either strictly pseudoconvex or convex at t^*, we obtain

$$\mathtt{M}_{\nabla\tilde{F}_i(t^*)}(\alpha^*)^T(\bar{t} - t^*) < 0 \quad \alpha^* \in [0, 1]. \tag{3.7}$$

Let $d = \bar{t} - t^*$. Since Ω is a convex set and $\bar{t}, t^* \in \Omega$, we have

$$t^* + \eta d = t^* + \eta(\bar{t} - t^*) = \eta\bar{t} + (1 - \eta)t^* \in \Omega, \quad \eta \in (0, 1).$$

According to Lemma 3.6, we obtain that $d \in D$, which means that

$$\mathtt{M}_{\nabla\tilde{g}_j(t^*)}(\alpha^*)^T d \leq 0, \quad \forall j \in J, \quad \alpha^* \in [0, 1]. \tag{3.8}$$

According to conditions (i) and (ii) of this Theorem, we obtain the following two new conditions: for $\alpha^* \in [0, 1]$,

(a) $\displaystyle\sum_{i=1}^{p} \lambda_i(\alpha^*)\mathtt{M}_{\nabla\tilde{F}_i(t^*)}(\alpha^*) + \sum_{j=1}^{m} \mu_j(\alpha^*)\mathtt{M}_{\nabla\tilde{g}_j(t^*)}(\alpha^*) = 0$ *for all* $\alpha^* \in [0, 1]$,

(b) $\mu_j(\alpha^*)\mathtt{M}_{\tilde{g}_j(t^*)}(\alpha^*) = 0$, for all $\alpha^* \in [0, 1]$ and $j = 1, 2, \cdots, m$.

Let A be the matrix of rows $\mathtt{M}_{\nabla\tilde{F}_i(t^*)}(\alpha^*)^T$ for $i = 1, 2, \cdots, p$ and C be the matrix of rows $\mathtt{M}_{\nabla\tilde{g}_j(t^*)}(\alpha^*)^T$ for $j \in J$. Now consider the two systems of Lemma 3.7. According to relations (3.7) and (3.8), we obtain that system I has a solution $d = \bar{t} - t^*$, and system II has no solution. This means that there do not exist multipliers $0 < \lambda_i \in \mathbb{R}$ for $i = 1, 2, \cdots, p$ and $0 \leq \mu_j \in \mathbb{R}$ for $j \in J$ such that

$$\sum_{i=1}^{p} \lambda_i \mathtt{M}_{\nabla\tilde{F}_i(t^*)}(\alpha^*) + \sum_{j \in J} \mu_j \mathtt{M}_{\nabla\tilde{g}_j(t^*)}(\alpha^*) = 0. \tag{3.9}$$

This contradicts conditions (a) and (b), since $\displaystyle\sum_{j \in J} \mu_j \mathtt{M}_{\nabla\tilde{g}_j(t^*)}(\alpha^*) = \sum_{j=1}^{m} \mu_j \mathtt{M}_{\nabla\tilde{g}_j(t^*)}(\alpha^*)$ with $\mu_j \mathtt{M}_{\tilde{g}_j(t^*)}(\alpha^*) = 0$ for $j = 1, 2, \cdots, m$. That is, $\mathtt{M}_{\tilde{g}_j(t^*)}(\alpha^*) \leq 0$ for $j \notin J$. This contradicts (i) and (ii) on the existence of multipliers $0 < \lambda_i \in \mathbb{R}$ for $i = 1, 2, \cdots, p$ and $0 \leq \mu_j \in \mathbb{R}$ for $j = 1, 2, \cdots, m$. Then we obtain that t^* is a Pareto optimal solution of problem (2.18). □

In particular, if there exists $\alpha^* \in [0, 1]$ such that conditions (i) and (ii) are satisfied, then t^* is a strongly Pareto optimal solution of problem (2.18).

Theorem 3.9. *Let Φ be the convex feasible set and $t^* \in \Phi$ be a feasible solution of problem (3.1). Suppose that the real-valued constraint functions $\mathtt{M}_{\tilde{g}_j(t)}(\alpha)$ are convex and continuously differentiable at t^* for $j = 1, 2, \cdots, m$ and that there exists a fuzzy-valued objective function, say the h-th fuzzy-valued objective function $\tilde{F}_h : \mathbb{R}^n \to \mathcal{N}(\mathbb{R})/\mathcal{S}$, such that $\tilde{F}_h$ is convex and continuously differentiable at t^* for $h = 1, 2, \cdots, p$. If there exist positive real-valued functions $\lambda(\alpha)$ defined on $[0, 1]$ and nonnegative real-valued functions $\mu_j(\alpha)$ defined on $[0, 1]$ for $j = 1, 2, \cdots, m$ such that*

(i) $\lambda(\alpha)\mathtt{M}_{\nabla \tilde{F}_h(t^*)}(\alpha) + \sum_{j=1}^{m} \mu_j(\alpha)\mathtt{M}_{\nabla \tilde{g}_j(t^*)}(\alpha) = 0$, *for all* $\alpha \in [0, 1]$,

(ii) $\mu_j(\alpha)\mathtt{M}_{\tilde{g}_j(t^*)}(\alpha) = 0$, *for all* $\alpha \in [0, 1]$ *and* $j = 1, 2, \ldots, m$,

then t^* is a weakly Pareto optimal solution of problem (2.18).

Proof. Suppose that t^* is not a weakly Pareto optimal solution. Then there exists $\bar{t} \in \chi$ such that $\tilde{F}_i(\bar{t}) \tilde{<} \tilde{F}_i(t^*)$ for $i = 1, 2, \cdots, p$. Thus we have

$$\mathtt{M}_{\tilde{F}_h(\bar{t})}(\alpha^*) < \mathtt{M}_{\tilde{F}_h(t^*)}(\alpha^*) \ for \ \alpha^* \in [0, 1].$$

Let f be a real-valued function defined by $f = \lambda(\alpha^*) \cdot \mathtt{M}_{\tilde{F}_h(t)}(\alpha^*)$ for $\alpha^* \in [0, 1]$. Then f is convex and continuously differentiable at t^*. We also have $f(\bar{t}) < f(t^*)$ since $\lambda(\alpha^*) > 0$. Using similar arguments to the proof of Theorem 3.5, conditions (i) and (ii) will be contradicted. Thus t^* is a weakly Pareto optimal solution of problem (2.18). □

4. Example

4.1. **Problem.** Consider the following nonlinear fuzzy multiobjective problem

$$(4.1) \quad \begin{cases} \max \tilde{f}_1(X) = (2, 3, 4, 5)x_1^2 + (1, 4, 6, 7)x_2^2 + (3, 4, 5, 6)x_3^2, \\ \max \tilde{f}_2(X) = (7, 8, 12, 13)x_1^2 + (13, 14, 16, 17)x_2^2 + (10, 11, 12, 13)x_3^2, \\ \text{s.t:} \\ (2, 3, 8, 9)x_1 + (6, 7, 8, 9)x_2 + (2, 3, 5, 8)x_3 \preceq (16, 18, 20, 22), \\ (7, 9, 12, 14)x_1 + (11, 12, 13, 15)x_2 + (14, 18, 19, 23)x_3 \preceq (26, 27, 30, 38), \\ x_i \geq 0, \quad (i = 1, 2, 3). \end{cases}$$

4.2. **Defuzzification.** By determining the medians of the α-cuts associated with each fuzzy number, we obtain the following deterministic problem

$$
(4.2)\quad \begin{cases}
\max \mathtt{M}_{\tilde{f}_1(X)}(\alpha) = \dfrac{1}{2}\left(7x_1^2 + (2\alpha+8)x_2^2 + 9x_3^2\right), \\
\max \mathtt{M}_{\tilde{f}_2(X)}(\alpha) = \dfrac{1}{2}\left(20x_1^2 + 30x_2^2 + 23x_3^2\right), \\
s.t: \\
11x_1 + 15x_2 + (-2\alpha+10)x_3 \leq 38, \\
21x_1 + (-\alpha+28)x_2 + 37x_3 \leq -7\alpha + 64, \\
\qquad \alpha \in [0,1], \\
x_i \geq 0, \quad (i = 1,2,3).
\end{cases}
$$

4.3. **Results.** For $w_1 = 0.5$ and $w_2 = 0.5$, we obtain the following table (Table 1) for different values of α ($\alpha \in [0,1]$).

Table 1. Different solutions for different values of α

α	X^*
1	(0.0000,2.1111,0.0000)
0.9	(0.0000,2.1291,0.0000)
0.8	(0.0000,2.1470,0.0000)
0.7	(0.0000,2.1648,0.0000)
0.6	(0.0000,2.1824,0.0000)
0.5	(0.0000,2.2000,0.0000)
0.4	(0.0000,2.2173,0.0000)
0.3	(0.0000,2.2346,0.0000)
0.2	(0.0000,2.2517,0.0000)
0.1	(0.0000,2.2688,0.0000)

The graph (Figure 2) below represents the Pareto front in the decision space for each value of α.

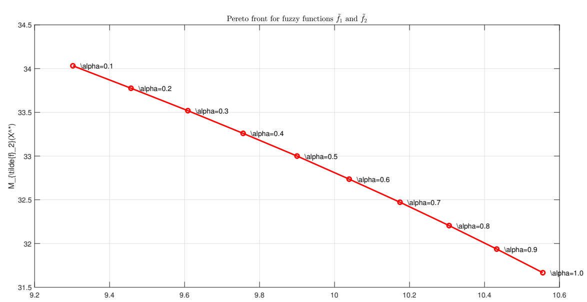

Figure 2. Pareto front

4.4. **Discutions.** The resulting Pareto front represents the set of efficient solutions to the multiobjective fuzzy optimization problem. In this problem, the functions to be maximized depend on a confidence level parameter α. Each point on the front corresponds to an optimal solution X^* obtained for a value of α, which illustrating a compromise between the two objectives: $\mathtt{M}_{\tilde{f}_1(X)}(\alpha)$ and $\mathtt{M}_{\tilde{f}_2(X)}(\alpha)$. We observe that when α decreases, i.e. when the decision maker accepts greater uncertainty (or fuzziness), the values of the two objective functions increase. In other words, the more "risk" the solution (low value of α), the more numerically advantageous it is. Conversely, a "safe" solution (with a high α) is more cautious but gives lower values of the objectives. The choice of a solution on the Pareto front therefore depends on the profile of the decision-maker and the importance he gives to each objective, represented by the weights w_1 and w_2 used in the weighted combination. If it prioritizes reliability, it will choose a high α and assign more weight to the objective it deems critical; if it accepts some level of fuzziness to achieve higher performance, it will opt for a lower value of α. Thus, the optimal solution chosen depends both on the preferences (weight) of the decision-maker and on his degree of acceptance of the risk linked to uncertainty.

5. Conclusion

In this work, we have developed a new theory to solve convex fuzzy multi-objective optimization problems within a quotient space of fuzzy numbers. Based on the notions of α-cuts, midpoint functions, and the Mare$\check{s}$ equivalence relation, we successfully extended the Karush-Kuhn-Tucker (KKT) optimality conditions originally formulated for single-objective fuzzy optimization to the multi-objective case. We introduced and rigorously defined fuzzy Pareto, weakly Pareto, and strongly Pareto optimality concepts, and derived necessary and sufficient KKT conditions associated with each. Our approach relies on a defuzzification process via midpoint functions, which allows the transformation of fuzzy problems into equivalent crisp optimization problems, enabling the application of classical convex analysis tools. The theoretical results confirm that this transformation preserves the Pareto optimality structure, thus offering a solid mathematical foundation for solving fuzzy optimization problems under uncertainty. Nevertheless, several limitations remain. The proposed theory is primarily suited to convex and differentiable fuzzy-valued functions; nonconvex or non-differentiable cases are not addressed and present opportunities for future exploration. Furthermore, one significant challenge is the difficulty in identifying the equivalence classes of fuzzy numbers, which are central to the quotient space construction. Lastly, the theoretical findings have not yet been illustrated through numerical examples or applied case studies. Future research will aim to generalize the theory to broader classes of fuzzy problems and explore its application to real-world scenarios in fields such as engineering, economics, and decision science.

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